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Conditional probabilistic approach in Fuzzy environment.

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Abstract—**Probability theory and fuzzy logic are quite distinct theoretical foundations for reasoning and decision making in situations of uncertainty. This paper establishes different approaches of fuzzy logic on probability and shows the fuzzy probabilistic application on real life uncertainty. Here we try to show fuzzy conditional probability and fuzzy Bayes' theorem and application of these two approaches in critical medical diagnosis in much more reliable manner.**

Keywords—**Crisp data, Fuzzy data, Triangular fuzzy number, α-cut, Bayes' theorem, Conditional probability, Fuzzy Arithmetic.**

I. INTRODUCTION

Fuzzy probability theory is an extension of probability theory for dealing with mixed probabilistic and non-probabilistic uncertainty. It provides a theoretical basis to model uncertainty which is only partly characterized by randomness and defies a pure probabilistic modeling with certainty due to a lack of trustworthiness or precision of the data or a lack of pertinent information. The fuzzy probabilistic model is settled between the probabilistic model and non-probabilistic uncertainty models. The significance of fuzzy probability theory lies in the treatment of the elements of a population not as crisp quantities but as set-valued quantities or granules in an imprecise manner, which largely complies with reality in most everyday situations. The development of fuzzy probability theory was initiated by H. Kwakernaak with the introduction of fuzzy random variables in 1978. Now it has vast applications in the field mathematics and computer science.

Through this article we are trying to reveal the advantages of fuzzy probabilistic approach, comparing to crisp probability and applying the fuzzy interval estimation on conditional probability and Bayes' theorem in imprecise manner to deal with more real life uncertain situations.

II. PRILIMINARY STUDY

Here we will discuss the fuzzy conditional probability and Bayes' theorem, depending on few important fuzzy approaches.

1. Fuzzy arithmetic approach
2. Implementation of triangular fuzzy number system.
3. Fuzzy probabilistic approach

1. Fuzzy arithmetic approach

Suppose A= $[a_1,a_2]$ and B= $[b_1,b_2]$ are two interval based fuzzy numbers.
Then

$$[a_1,a_2] + [b_1,b_2] = [a_1+b_1,a_2+b_2]$$
$$[a_1,a_2] - [b_1,b_2] = [a_1-b_2, a_2-b_1]$$
$$[a_1,a_2] . [b_1,b_2] = [a_1.b_1,a_2.b_2]$$
$$[a_1,a_2] / [b_1,b_2] = [a_1/b_2, a_2/b_1]$$

2. *Triangular fuzzy number*

A triangular fuzzy number **a** is defined by a triplet (a_1, a_2, a_3). The membership function is defined as

$$\mu_a (x) = \{ (x - a_1) / (a_2 - a_1) \text{ if } a_1 \le x \le a_2$$
$$(a_3 - x) / (a_3 - a_2) \text{ if } a_2 \le x \le a_3$$
$$0 \text{ otherwise}\}$$

Where μ_a (x) is membership value of the fuzzy number.

The triangular fuzzy number is based on three-value judgment: The minimum possible value a_1, the most possible value a_2 and the maximum possible value a_3.

3. Fuzzy probabilistic approach

Let *A* and *B* be (crisp) subsets of $X = \{x_1, ..., x_n\}$. We have a discrete (finite) fuzzy probability distribution

$P(\{x_i\}) = a_1,\ 0 < ai < 1,\ 1 \le i \le n.$

Suppose $A = \{x_1, ..., x_k\}$, $1 \le k < n$, then define

$$P(A)[\alpha] = \{\sum_{i=1}^{k} a_i \mid S\},$$

for $0 \le \alpha \le 1$, where **S** stands for the statement " $a_i \in a_i[\alpha]$, $1 \le i \le n$,

$\sum_{i=1}^{n} a_i = 1$ *This is restricted fuzzy arithmetic. We now* show that the P(A)[α] are the α-cuts of a fuzzy number $P(A)$. But first we require some definitions.
Define

$$S = \{(x_1, ..., x_n) | x_i \ge 0 \text{ all } i, \sum_{i=1}^{n} x_i = 1\}$$

and then also define $Dom[\alpha] = (\prod_{i=1}^{n} a_i[\alpha]) \cap S)$ *for* $0 \le \alpha \le 1$.

If $P(A)$ is a fuzzy number ,we can express it with fuzzy interval estimation as $P(A)[\alpha] = [pa_1(\alpha), pa_2(\alpha)]$.

III. FUZZY CONDITIONAL PROBABILTY

Let $A = \{x_1, ..., x_k\}$, $B = \{x_l, ..., x_m\}$ for $1 \le l \le k \le m \le n$ so that A and B are not disjoint. We wish to define the fuzzy conditional probability of A given B. We will write this fuzzy conditional probability as $P(A|B)$. We now present two definitions for fuzzy conditional probability and then argue in favor of the first definition.

Our first definition is

$$P(A|B)[\alpha] = \left\{ \frac{\sum_{i=l}^{k} a_i}{\sum_{i=l}^{m} a_i} \mid S \right\}$$

In this definition the numerator of the quotient is the sum of the a_i in the intersection of A and B, while the denominator is the sum of the a_i in B.

Our second definition is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

The first definition always produces a fuzzy probability in [0,1].

Example:
Let $n = 4$, $A = \{x_1, x_2\}$, $B = \{x_2, x_3\}$ and all the a_i are uncertain with $a_1 = (0.1/0.2/0.3)$, $a_2 = (0.2/0.3/0.4)$, $a_3 = (0/0.1/0.2)$ and $a_4 = (0.3/0.4/0.5)$.

A. Application of second definition

Since $A \cap B = \{x_2\}$ we find that
$P(A \cap B) = a_2$.= (0.2/0.3/0.4)
As $B = \{x_2, x_3\}$, $P(B) = (0.2/0.4/0.6)$.
So,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{(0.2/0.3/0.4)}{(0.2/0.4/0.6).}$$

Whose $\alpha = 0$ cut is the interval [1/3 , 2] with right end point greater than one. In probability, more than one is not acceptable. We often get these types of undesired result through second definition. So we adopt the first definition of fuzzy conditional probability.

B. Application of first definition

According to the definition of first fuzzy conditional probability

$$P(A|B)[\alpha] = \left\{ \frac{a_2}{a_2 + a_3} \mid S \right\}$$

Using triangular fuzzy number we get

$a_2 = [\,0.1\,\alpha + 0.2\,,\ 0.4 - 0.1\,\alpha\,]$
$a_3 = [\,0.1\,\alpha\,,\ 0.2 - 0.1\,\alpha\,]$

$a_2 + a_3 = [\,0.1\,\alpha + 0.2\,,\ 0.4 - 0.1\,\alpha\,] + [\,0.1\,\alpha\,, 0.2 - 0.1\,\alpha\,]$
$= [0.4\,,\,0.4\,]$

$$P(A|B)[\alpha] = \frac{[\,0.1\,\alpha + 0.2\,,\ 0.4 - 0.1\,\alpha\,]}{[0.4,\ 0.4]}$$

$$= [0.5 + 0.25\alpha,\ 1 - 0.25\alpha]$$

$P(A|B)[0] = (\,0.5,1)$ and $P(A|B)[1] = (0.75,0.75)$

So $P(A|B)$=(0.5 / 0.75 / 1) is a triangular fuzzy number.

IV. FUZZY BAYES' THEOREM

A. Normal crisp Bayes' theorem

Let Ai, $1 \le i \le m$, be a partition of $X = \{x_1, ..., x_n\}$. That is, the A_i is non-empty, mutually disjoint and their union is X, where X is universal set.
We do not know the probability of the Ai but we do know the conditional probability of Ai given the state of nature. There is a finite set of chance events, also called the states of nature,
$S = \{S_1, ..., Sn\}$ over which we have no control.

What we do know is $a_{ik} = P(A_i|S_k)$,
for $1 \le i \le m$ and $1 \le k \le n$,. If the operative state of nature is S_k, then the a_{ik} give the probabilities of the events Ai.

The probability that the state of nature Sk is in force, given the information that outcome Aj has occurred, is given by Bayes' formula

$$P(S_k|Aj) = \frac{P(A_j|S_k)P(S_k)}{\sum_{k=1}^{n} P(A_j|S_k)P(S_k)}$$

for $1 \le k \le n$. The $a_{kj} = P(S_k|A_j)$, $1 \le k \le n$, is the posterior probability distribution over the states of nature.

B. Fuzzy Bayes' theorem

a.

From an example we can explain fuzzy Bayes' theorem in a better way.

Let there be only two states of nature $S1$ and $S2$ with fuzzy prior probabilities.

$p_1 = P(S_1) = (0.3/0.4/0.5)$ and $p_2 = P(S_2) = (0.5/0.6/0.7)$. There are also only two events A_1 and A_2 in the partition of X with known conditional probabilities
$p_{11} = P(A_1|S_1) = 0.2$, $p_{21} = P(A_2|S_1) = 0.8$, $p_{12} = P(A_1|S_2) = 0.7$ and $p_{22} = P(A_2|S_2) = 0.3$.

Using fuzzy interval estimation we can derive $P(S_1|A_1)[\alpha]$ and $P(S_2|A_1)[\alpha]$.

$P(S_1) = p_1[\alpha] = [\, 0.1\,\alpha + 0.3\,,\; 0.5 - 0.1\,\alpha\,] = [p_{11}, p_{12}]$
$P(S_2) = p_2[\alpha] = [\, 0.1\,\alpha + 0.5\,,\; 0.7 - 0.1\,\alpha\,] = [p_{21}, p_{22}]$

$$P(S_1|A_1)[\alpha] = \frac{P(S_1)\,P(A_1|S_1)}{P(S_1)\,P(A_1|S_1) + P(S_2)\,P(A_1|S_2)}$$

This Bayes' equation can be represented through interval estimation with α-*cut* relation.

Where **S** is "$p_i \in p_i[\alpha]$, $i = 1,2$ and $p_1 + p_2 = 1$". Both α-cuts are easily found.

Let $p_i[\alpha] = [p_{i1}(\alpha), p_{i2}(\alpha)]$, for $i = 1, 2$.

Then fuzzy representation is

$$P(S1|A1)[\alpha] = \left[\frac{0.2p_{11}(\alpha)}{0.2p_{11}(\alpha) + 0.7p_{22}(\alpha)}, \frac{0.2p_{12}(\alpha)}{0.2p_{12}(\alpha) + 0.7p_{21}(\alpha)}\right]$$

Same way the fuzzy representation of $P(S_2|A_1)[\alpha]$ is

$$P(S_2|A_1)[\alpha] = \left[\frac{0.7p_{21}(\alpha)}{0.2p_{12}(\alpha) + 0.7p_{21}(\alpha)}, \frac{0.7p_{22}(\alpha)}{0.2p_{11}(\alpha) + 0.7p_{22}(\alpha)}\right]$$

So,

$$P(S_1|A_1)[\alpha] = \left[\frac{0.02\,\alpha + 0.06}{0.55 - 0.05\,\alpha}, \frac{0.10 - 0.02\,\alpha}{0.45 + 0.05\,\alpha}\right]$$

And

$$P(S_2|A_1)[\alpha] = \left[\frac{0.07\,\alpha + 0.35}{0.45 + 0.05\,\alpha}, \frac{0.49 - 0.07\,\alpha}{0.545 - 0.05\,\alpha}\right]$$

If we take α *as* 0 and 1 then $P(S_1|A_1)[\alpha]$ and $P(S_2|A_1)[\alpha]$ can be represented as triangular fuzzy number.

Now we may compute $P(A1)$ using the fuzzy posterior probabilities. It has α-cuts

$(0.2)P(S_1|A_1)[\alpha] + (0.7)P(S_2|A_1)[\alpha]$.

Case study on Testing for HIV

$A1$= the event that a person is infected with the HIV virus.
$A2$= event that a person is not infected.
$B1$= event that the test T is "positive" indicating that the person has the Virus.
$B2$= event that the test T gives the result of "negative", or the person does not have the virus.

We want to find the conditional probability of A_1 given B_1, or $P(A_1|B_1)$.

We take a random sample to estimate the probabilities
$p_{11} = P(A1 \cap B1)$, $p_{12} = P(A1 \cap B2)$, $p_{21} = P(A2 \cap B1)$ and $p_{22} = P(A2 \cap B2)$.

Assume we obtain the estimates

$p_{11} = 0.095$, $p_{12} = 0.005$, $p_{21} = 0.045$ and $p_{22} = 0.855$.

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To show the uncertainty in these point estimates we now substitute fuzzy numbers for the p_{ij}.

Let p_{11} = (0.092 / 0.095 / 0.098), p_{12} = (0.002 /0.005 / 0.008),
p_{21} = (0.042 / 0.045 / 0.048) and p_{22} = (0.825 / 0.855 / 0.885).

According to the first definition of fuzzy conditional probability,

$$P(A_1|B_1)[\alpha] = \left\{ \frac{p_{11}}{p_{11} + p_{21}} \quad | \, S \right\}$$

$p_{11}[\alpha]$ = [0.003 α + *0.092* , *0.098 - 0.003* α]
$p_{21}[\alpha]$ = [0.003 α + *0.042* , *0.048 - 0.003* α]

$$P(A_1|B_1)[\alpha] = \frac{[0.003\,\alpha + 0.092 ,\ 0.098 - 0.003\,\alpha]}{[0.14, 0.14]}$$

$$=\left[\frac{0.003\,\alpha + 0.092}{0.14} , \frac{0.098 - 0.003\,\alpha}{0.14} \right]$$

The probability that "a person infected with the HIV virus when test *T* is "positive" can be represented within a range in presence of interval estimation. Using normal crisp conditional probability we can test a person is infected with the HIV virus when test *T* is "positive", but it will generate only one probable result. But in real life application due to uncertain environment it is much more reliable to generate the probable results within a range of interval.

Here

$$=\left[\frac{0.003\,\alpha + 0.092}{0.14} , \frac{0.098 - 0.003\,\alpha}{0.14} \right]$$

gives all possible probabilistic values within lower and upper limit of the fuzzy interval.

Now for $P(A_1|B_1)[0]$ = [92/140 , 95/140] and
$P(A_1|B_1)[1]$ = [92/140 , 95/140]

So, $P(A_1|B_1)$= (92/140 / 95/140 / 98/140) is a triangular fuzzy representation of a person's infection with the HIV virus when test *T* is "positive".

V. DIAGRAMATIC REPRENTATION

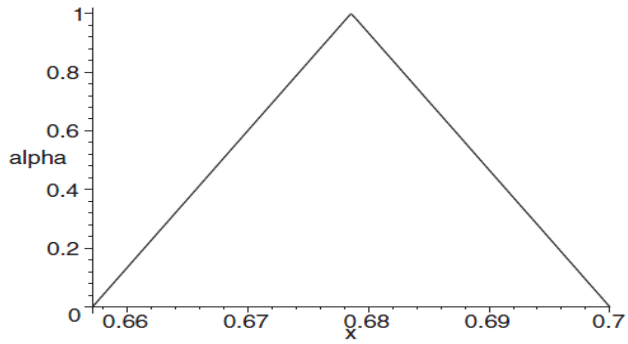

Fig-1: Fuzzy Probability of HIV Given Test Positive

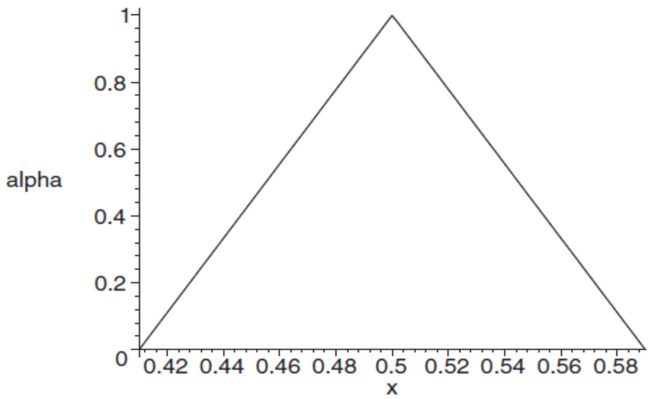

Fig-2: $P(A_1)$ Using the Fuzzy Prior

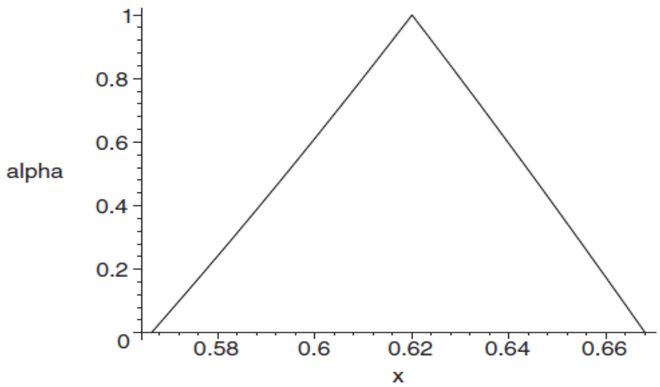

Fig-3: $P(A_1)$ Using the Fuzzy Posterior

VI. CONCLUSION

In this paper, it is proposed that, in presence of fuzzy environment, probabilistic decisions making can be much more realistic and can solve real life uncertain situations, using interval estimation technique on imprecise data. Through this approach we can solve different problems of medical field, economic sector, agricultural areas etc.

VII. FUTURE ENHANCEMENT

In future we can work with discrete fuzzy random variable to deal with Fuzzy Binomial, Fuzzy Poisson distribution. Theses can give high impact on the following areas.
(1) using the fuzzy Poisson to approximate values of the fuzzy binomial.
(2) using the fuzzy binomial to calculate the fuzzy probabilities of "overbooking".
(3) using the fuzzy Poisson to estimate the size of a rapid response team to terrorist attacks.

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